# LOCKING PHENOMENA IN PHASE LOCKED OSCILLATORS

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**ABSTRACT**. The operation of typical automatic phase control circuits has been studied with particular reference to locking range and time. The similarity of an injection synchronised oscillator with a slow acting amplitude stabilisation circuit to a standard APC system is pointed out. The pull-in phenomenon in APC circuits, with either sinusoidal or triangular comparators, incorporating low-pass filters in the feedback loop has been analysed. Expressions have been derived for the locking range and time of APC circuits using a sinewave comparator. Experimental results obtained for such circuits have been presented and are found to be in good agreement with the computed values.

## INTRODUCTION

A phase locked oscillator, as its name implies, is an oscillator the phase of which is locked to an input reference oscillation. It operates by detecting the phase difference between the two oscillations and controlling the frequency of the oscillator in correspondence to a measure of this phase difference after suitable filtering.

A phase locked oscillator is essentially a feedback device incorporating a narrow-band filter. Because of the narrow band feedback process it reduces internally generated noises and disturbances as well as disturbances appearing at the input. A phase-locked oscillator thus finds uses in noise- and jitter-free frequency synthesis and in frequency tracking. An *APC* circuit has a close similarity to injection synchronised oscillator where an oscillation of a desired frequency is injected into the oscillator. The amplitude and frequency of the injected voltage must be such as to quench the free oscillation, the quenching action being obtained through an instantaneous limiter type non-linearity which attenuates the weaker signal more than the stronger.

Because of the close similarity between the phase locked oscillator and the injection synchronised oscillator, this paper deals with some of the characteristics, such as the locking range and time, of both types of oscillator.

In section 2, a simple explanation of single frequency synchronisation phenomena is given and it is emphasised that it is imperative to use a limiter type charac-

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teristic. This is followed by a study in section 3 of the effect of low frequency time constant of the gain control arrangement of an injection synchronised oscillator. The similarity between this type of oscillator and a phase locked oscillator having a low pass filter in the loop has been pointed out.

The behaviour of a simple *APC* circuit with sinusoidal and linear phase comparators has been analysed in section 4. An explanation of pull-in effect is given in section 5 for *APC* loops with two different filters. From the accompanying analysis an approximate idea of the pulling range and time can be formed.

Expressions for locking time and range have been derived in section 6. Two types of filters have been considered in this connection—one with negligible high frequency gain and the other with finite high frequency gain. Simple but approximate relations for the locking range and time have been given. The practical results with regard to locking range in *APC* circuits with typical filter networks in the loop have been presented in section 7. These are in good agreement with the theoretical formulae.

## SINGLE FREQUENCY DIRECT SYNCHRONISATION PHENOMENA

In this section, single frequency direct synchronisation in an oscillator of the type shown in Fig. 1 will be considered.

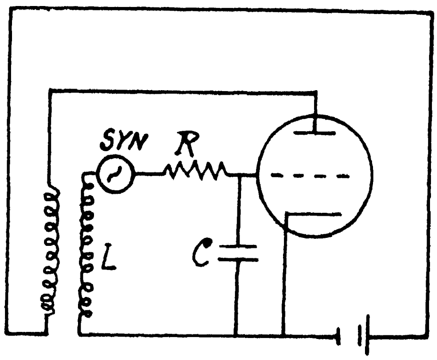


Fig. 1. Schematic diagram of a directly synchronised grid tuned oscillator.

The phenomenon of synchronisation in injection synchronised oscillators can be understood in the following way. Due to the mixing process provided by the inherent non-linearity between the grid voltage and the plate current in the tube circuit, the effective transconductance of the tube or the gain parameter has an in-phase as well as a quadrature component. The in-phase component modulates the amplitude of the free running oscillator and the quadrature component modulates the frequency of the oscillator. The magnitude of these quantities will depend on the relative amplitudes and the phase difference between the reference signal and the free running oscillator. If the frequency difference $\Omega/2\pi$ between these signals is not large and the synchronising amplitude is adequate, then it may be expected that the phase difference will attain a steady value $\Phi_s$ and the phase of the free running oscillator will be locked in synchronism with that of the reference signal. The amplitude of the synchronising signal required for synchronising the free running oscillator depends not only on the difference of frequency between the free running oscillator and the reference signal but also on the amplitude of the free running oscillation.

Locking can also be considered as selection of the external signal and supression of the internal. The suppression depends on the fact that when two non-coherent signals are applied to a limiter-type non-linear transference, the weaker signal is attenuated more than the stronger. One can understand the mechanism of weaker signal suppression by a reference to Fig. 2 which depicts relative transmission at two frequencies through a cubic type non-linearity,

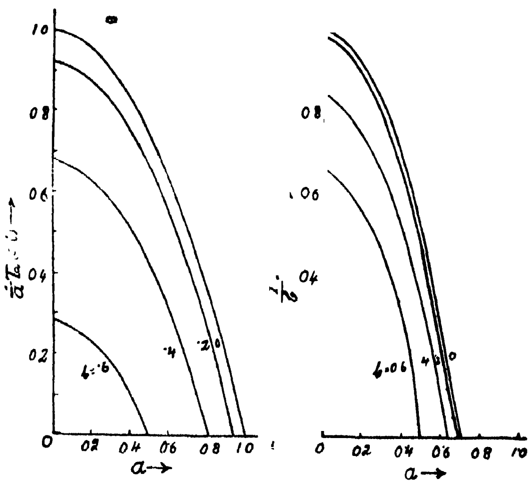

Fig. 2. Transmission characteristics of a limiter type non-linearity.

$$x_{out} = a_1 x_{in} - a_3 x_{in}^3. \qquad \ldots \quad (2.1)$$

If we assume that $x_{in}$ is of the form

$$x_{in} = a \cos \psi_1 + b \cos \psi_2 , \qquad \ldots \quad (2.2)$$

then the transmission of component at frequency $\omega_1 = d\psi_1/dt$, having the amplitude '$a$' in presence of the another component at frequency $\omega_2$ and of amplitude '$b$' is

$$T_a(a, b) = [1 - 3/4(a^2 + 2b^2)]a. \qquad \ldots \quad (2.3)$$

Similarly, for the component at frequency $\omega_2$,

$$T_b(a, b) = b\left[1 - \frac{3}{4}(b^2 + 2a^2)\right]. \qquad \ldots \quad (2.4)$$

On the other hand, if we assume an expansion-type characteristic such as given by

$$x_{out} = \exp(x_{in}) - \exp(-x_{in}), \quad \dots \quad (2.5)$$

then the transmissions are given by

$$T_a(a, b) = 2I_1(a)I_0(b), \quad \dots \quad (2.6)$$

and

$$T_b(a, b) = 2I_0(a)\, I_1(b). \quad \dots \quad (2.7)$$

A plot of $T_a(a, b)/a$ is shown in Fig. 3 from which one easily understands that this type of characteristic helps a weaker signal to build up.

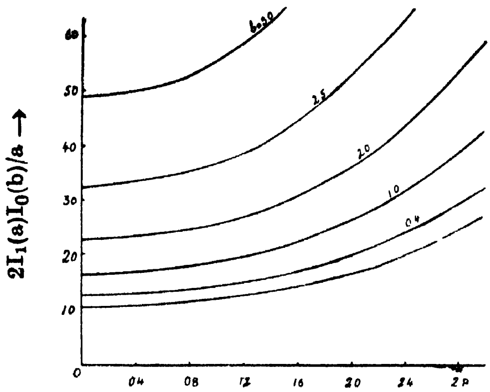


Fig. 3. Transmission characteristics of an expandor type non-linearity.

It is evident, then, that a limiter type characteristic is preferable to an expandor type characteristic so far as the interference reduction is concerned. In the case of limiter type characteristics, if we consider the same frequency synchronisation phenomena, it can be easily shown that the locking equation is given by,

$$\frac{d\Phi}{dt} = \Omega - K \sin \Phi, \quad (2.8)$$

where $K$ is a constant which depends on '$a$' and '$b$' and $\Omega$ is the initial frequency difference. It is to be noted that the process of synchronisation is due to instantaneous limiting and thus no filtering other than at r.f. is possible. In fact the value of $K$ is given approximately by $K = \frac{a}{b} \cdot \frac{\omega_0}{2Q}$ for single tuned circuit.

It is to be noted that if there are finite transmissions at other frequencies generated through the process of limiting then the statements made earlier in connection with interference reduction do not apply. If, for example, the component at frequency $(2\omega_1 - \omega_2)$ has finite transmission then it can be shown that the amount of suppression obtainable is small. This is due to the fact that the input to the limiter in such a case shows too little amplitude modulation for limiting to be effective. In any event the oscillator should have a soft characteristic, that is, it should be a free oscillator even in the absence of any r.f. input excitation.

Thus oscillators working in any subharmonic mode should be free-running and not of the regenerative divider type.

## EFFECT OF LOW-FREQUENCY TIME CONSTANT OF THE GAIN CONTROL ARRANGEMENT ON THE PHASE EQUATION

In the previous case we have assumed instantaneous limiting due to which there was 'strong signal capture' and 'small signal rejection'. We shall now study the case when the limiting is not instantaneous. In most of the practical oscillator circuits the gain control arrangement is only partly instantaneous. For example, the R-C time constant in the self-bias circuit of an oscillator provides a slow-acting gain control circuit.

Let us suppose that the gain is controlled by the rectified envelope and the time constant of the circuit is $T$ secs. Then the open loop equations can be written as (Fig. 4),

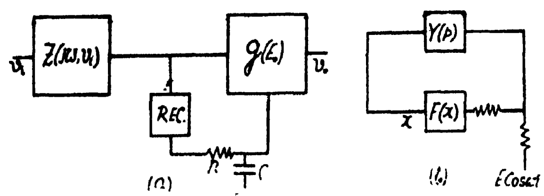

Fig. 4. Equivalent block diagram of a directly synchronised oscillator with a filter in the gain control circuit.

$$v_0 = Z(j\omega, v_i)\ g(E_0)v_i \qquad \dots \quad (3.1)$$

and

$$T\frac{dE_0}{dt} = E_i - E_0 = av_i - E_0. \qquad \dots \quad (3.2)$$

where $v_i$ and $v_0$ are the input and output voltages respectively and $E_i$ is the detected voltage and $E_0$ is the control voltage.

Now let us suppose that $v_i$ consists of two components one at an angular frequency $\omega_1$ and the other at the angular frequency $\omega_2$ of amplitudes '$a$' and '$b$' respectively. Then

$$v_i = a\cos\psi_1 + b\cos\psi_2. \qquad \dots \quad (3.3)$$

It can be shown that if the control characteristic is given by $g(E_0) \simeq (1-\beta/E_0/^2)$ $g_0$, the net transmissions at frequencies $\omega_1$ and $\omega_2$ are respectively

$$T_a(a, b) = a[1 - g_0(a^2 + b^2 + b^2 G_1 \cos\Phi_1)], \qquad \dots \quad (3.4)$$

and

$$T_b(a, b) = b[1 - g_0(a^2 + b^2 + a^2 G_1 \cos\phi_1)], \qquad \dots \quad (3.5)$$

where $G_1(\Delta\omega)\exp[j\Phi_1(\Delta\omega)]$ represents the filter transmission at the difference frepuency $\Delta\omega = \omega_1 \sim \omega_2$. Hence from the above it is evident that if $G_1 \cos\Phi_1 \neq 0$, then there will be non-linear discrimination of one frequency against the another.

Similar conclusions apply for other types of gain control characteristics, like $g(E_0) = g_0(1-\beta|E_0|)$.

Now for the loop shown in Fig. 4 the component '$a$' at $\omega_1$ should satisfy the following eq.

$$a \cos(\omega t+\Phi) \left[ F_a(a, b)+\frac{E}{a} \cos \Phi \right]+E \sin \Phi \sin(\omega_1 t+\Phi)$$

$$\frac{}{y(p)} \cdot \cos(\omega_1 t+\Phi), \tag{3.6}$$

where

$$y(p) = \frac{\alpha p}{p^2+\alpha p+\omega_0^2} \tag{3.7}$$

and $E\cos(\omega_1 t+\Phi)$ is the external synchronising voltage. Now

$$\frac{1}{y(p)} = 1+\frac{1}{\alpha}\left(p+\frac{\omega_0^2}{p}\right) \simeq 1+\frac{2}{\alpha}S+j\frac{\omega_1^2-\omega_0^2}{2\omega_1}, \tag{3.8}$$

If we put $\quad p = j\omega_1+S$

where $S$ represents an operator in a slow time scale. The amplitude and phaes equations are approximately given by

$$\frac{2}{a} \cdot \frac{da}{dt} = a\left[F_a(a, b)-1\right] +E\cos\phi, \quad \ldots \tag{3.9}$$

$$\frac{2}{\alpha} \quad \frac{d\phi}{dt} \qquad \frac{\omega_1^2-\omega_0^2}{\alpha\omega_1} - \frac{E}{} \sin\phi. \tag{3.10}$$

Hence from (3.2) and (3.10) we have,

$$\frac{2}{\alpha} \cdot \frac{d\phi}{dt} = -\frac{\omega_1^2-\omega_0^2}{\alpha\omega_1} - \frac{\alpha_0}{E_0} \cdot \frac{E}{(1+PT)} \sin\phi, \tag{3.11}$$

which can be rewritten in the form

$$\frac{d\phi}{dt} = \Omega - G(p)\frac{\omega_0}{2Q} \cdot \frac{E}{E_0}\sin\phi, \quad \ldots \tag{3.12}$$

where $G(p)$ represents the transfer function of a simple $R$-$C$ lag filter. It is to be noted that Eq. (3.15) represents the APC equation with a simple $R$-$C$ lag filter (see Sec. 5, 2). It is apparents that the maximum rate of the input frequency variation depends on the $Q$ of the tuned circuit and the time constant of the gain control circuit. Experimental results for locking range for different values of the $RC$ product in the grid circuit of an oscillator are shown in Fig. 4(a). These confirm the theoretical findings.

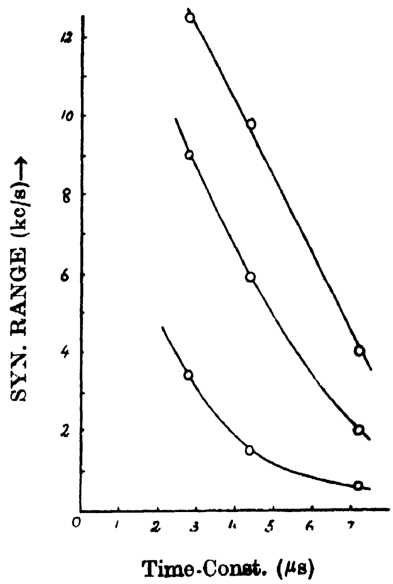


Fig. 4(a). Pull-in performance of the directly synchronised oscillator with a slow acting gain control arrangement. The variation of the locking range with the detector time constant is shown for three values of the synchromising voltage.

## THE SIMPLE APC CIRCUIT

In this section we shall consider the simple APC circuit shown in Fig. 5. It contains an oscillator whose output frequency is approximately equal to the desired

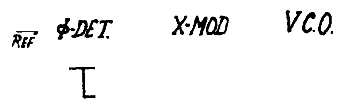


Fig. 5. Block diagram of a standard APC circuit.

frequency, a phase detector and a reactance modulator. The phase detector performs the function of an error detecting device in the sense that it detects the error between the instantaneous phases of the reference input and the oscillator.

APC loops can be classified into different categories depending on the nature of the signals and the type of comparator used. The signal may be continuous or interrupted. The comparator may be linear or sinusoidal. In the following analysis we will assume the signal to be continuous.

### 4.1. *APC Circuit with Sinusoidal Comparator* :

We shall here analyse the behaviour of an APC circuit with a sinusoidal comparator. In this case the output of the phase detector $\delta E$ is

$$\delta E = \mu \sin\phi, \tag{4.1}$$

where $\phi$ is the instantaneous phase difference between the signals and $\mu$ represents the gain of the phase detector in volts per radian. This outout voltage tends to keep the output freauency of the oscillator constant through the reactance modulator, the sensitivity of which can be represented by

$$\beta = \delta\omega/\delta E. \tag{4.2}$$

Thus $\beta$ represents the gain of the modulator in rad/sec per volt. Then the governing equation of the system is given by

$$\frac{d\phi}{dt} \quad \Omega - K \sin\phi, \qquad \ldots \tag{4.3}$$

where $K = \mu\beta$. If $\Omega$, the undisturbed beat angular frequency, is less than $K$, then there is a fixed value $\phi = \phi_s$ for which $d\phi/dt = 0$, so that

$$\phi_s = \sin^{-1}(\Omega/K). \tag{4.4}$$

Thus if the initial detuning lies within the limiting synchronisation range ($K$) then the phase of the output oscillation will be locked in phase in synchronism with the reference input. Solving Eq. (4.3), expressions for the instantaneous phase and frequency of the oscillator can be readily obtained.

If $\Omega > K$ then the instantaneous phase and frequency are given by

$$\phi = 2\tan^{-1}\left[\frac{1}{\Omega/K} + \frac{\sqrt{(\Omega/K)^2-1}}{\Omega/K} \quad \tan K\frac{\sqrt{(\Omega/K)^2-1}}{2}(t+t_o)\right] \ldots \tag{4.5a}$$

and

$$\frac{d\phi}{dt} = K\left[\left(\frac{\Omega}{K}\right)^2 -1\right]\frac{1}{(\Omega/K)^2+\cos(2\psi-\beta_0)} \tag{4.5b}$$

where

$$\psi = K\frac{\sqrt{(\Omega/K)^2-1}}{2}(t+t_0)$$

and

$$\beta_0 = \tan^{-1}\sqrt{\left(\frac{\Omega}{K}\right)^2-1}.$$

Inside the pull-in range (i.e. $\Omega < K$) the expression for the instantaneous phase is

$$\phi = 2\tan^{-1}\left[\frac{1}{\Omega/K} - \frac{\sqrt{1-(\Omega/K)^2}}{\Omega/K} \tanh K\frac{\sqrt{1-(\Omega/K)^2}}{2}(t+t_0)\right] \ldots \tag{4.6}$$

A Fourier analysis of Eq. (4.5b) yields the values of the spectral components of the instantaneous frequency

$$\frac{d\phi}{dt} = K\sqrt{\left(\frac{\Omega}{K}\right)^2-1}\ [1-2r\cos(2\psi-\beta_0)+2r^2\cos 2(2\psi-\beta_0)-$$
$$\ldots+(-1)^n 2r^n\cos n(2\psi-\beta_0)], \qquad \ldots \tag{4.7}$$

where $$r = \frac{\Omega}{K} - \sqrt{\left(\frac{\Omega}{K}\right)^2 - 1}\,.$$

It is clear from the study of Eq. (4.7) that the instantaneous frequency will show violent fluctuation as $\Omega$ approaches $K$ and the output will contain a large number of spectral components of significant amplitudes separated by the multiples of the beat frequency. The instantaneous phase for frequencies away from the cross-over (i.e. $\Omega >> K$) can be represented by the following approximate relation

$$\phi = \omega t + \alpha + m_1 \sin \omega t + m_2 \sin (2\omega t + \theta) \qquad \ldots \quad (4.8)$$

The values of $\omega$, $\alpha$, $m_1$, $m_2$ and $\theta$ can be determined by substituting the value of $\phi$ given by Eq. 4.8 into Eq. 4.3 and equating components at different frequencies.

4.2 *Linear Phase Comparator* :

We shall now consider the case where the instantaneous detected voltage is a symmetrical triangular function of the phase difference (Fig. 6). The loop equation now becomes

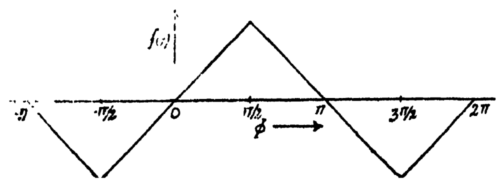


**Fig. 6. Output of the linear phase comparator.**

$$\frac{d\phi}{dt} = \Omega - K\phi, \quad -\pi/2 \leqslant \phi \leqslant +\pi/2 \qquad \ldots \quad (4.9a)$$

and $$\frac{d\phi}{dt} = \Omega - K(\pi - \phi), +\pi/2 \leqslant \phi \leqslant 3\pi/2. \qquad \ldots \quad (4.9b)$$

It may be mentioned that a symmetrical triangular comparator can be realised in practice by mixing two strongly limited r.f. voltages.

The period of the instantaneous beat frequency can be found by solving (4.9) and is given by

$$T = \frac{2}{K} \log_e \left(\frac{1+x}{1-x}\right), \qquad \ldots \quad (4.10)$$

where $$x = \frac{K\pi}{2\Omega}$$

If the comparator characteristic is triangular but asymmetric, i.e., the detected voltage is

$$V = K_1\phi, \; -\phi_1 \leqslant \phi \leqslant \phi_1$$

and ... $$V = K_2\phi, (\pi - \phi_1) \leqslant \phi \leqslant (\phi_1 + \pi)$$

and the corresponding expression for a beat period is

$$T = \frac{1}{K_1} \log_e \left[ \frac{1+\frac{K_1\phi_1}{\Omega}}{1-\frac{K_1\phi_1}{\Omega}} \right] + \frac{1}{K_2} \log_e \left[ \frac{1+K_2\frac{\pi+\phi_1}{\Omega}}{1-K_2\frac{\pi-\phi_1}{\Omega}} \right] \tag{4.11}$$

The instantaneous beat frequency $2\pi/T$ and also the discriminator output voltage for the case of a symmetrical triangular comparator have been plotted against $\Omega T$ and compared with those of the case when the phase comparator is sinusoidal (Fig. 7). It will be observed that there is a close functional similarity between

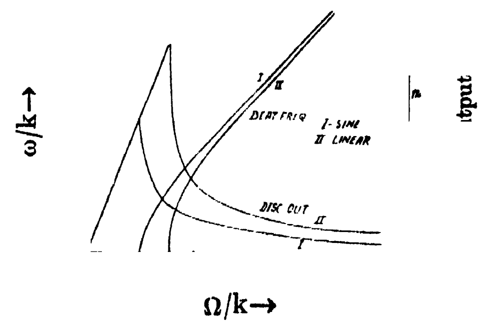


Fig. 7. Discriminator output and beat frequency vs. error characteristics.

an APC loop with a symmetrical triangular comparator and that with a sinusoidal comparator, although the range in the former is larger by a factor of $\pi/2$.

## APC LOOP WITH FILTER

It will be observed from the analysis in section 4 that for APC circuits without a filter incorporating either type of comparator the value of the locking ratio ($\Omega/K$) determines completely their performance characteristics, viz., locking range, noise band-width and the nature of the response to transients of phase and frequency. Now it is considered desirable to be able to control these parameters independently of one another, in particular, to reduce the noise band-width. Incorporation of an appropriate low pass filter in the loop helps to achieve this. Inclusion of the filter network will, however, introduce the so-called pull-in phenomenon. The response to transients may no longer be deadbeat even if the difference frequency lies within the locking range and the instantaneous frequency may drift for a few beat cycles of continuously decreasing frequency till equillibrium is reached.

### 5.1. *Pull-in Effect in an APC Circuit with Filter and Symmetrical Triangular Comparator* :

We shall now consider the pull-in phenomena in an APC circuit with symmetrical triangular phase comparator and a low pass filter. Introduction of a filter in the feedback loop modifies Eq. (4.9) to the following equations :

$$\frac{d\phi}{dt} = \Omega - K\cdot G(p)\phi, \quad -\pi/2 \leqslant \phi \leqslant +\pi/2 \qquad \ldots \quad (5.1)$$

and

$$\frac{d\phi}{dt} = \Omega - K\cdot G(p)(\pi-\phi), \quad +\pi/2 \leqslant \phi \leqslant 3\pi/2 \qquad (5.2)$$

where $G(p)$ stands for the filter-transfer function. For the filter network shown in Fig. 8(a) we have from equations (5.1) and (5.2)

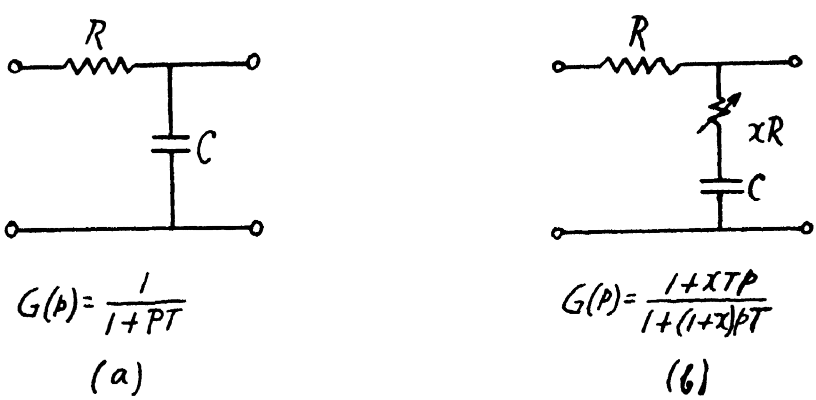


Fig. 8. Simple low pass filter :
(*a*) with negligible high frequency gain (*b*) with finite high frequency gain.

$$\frac{d^2\phi_1}{dt^2} + \frac{1}{\alpha}\cdot\frac{d\phi_1}{dt} + \frac{K}{\alpha}\cdot\phi_1 = 0, \quad -\pi/2 \leqslant \phi \leqslant \pi/2 \qquad \ldots \quad (5.3)$$

where

$$\phi_1 = \phi - \Omega/K$$

and

$$\frac{d^2\phi_2}{dt^2} + \frac{1}{\alpha}\cdot\frac{d\phi_2}{dt} - \frac{K}{\alpha}\phi_2 = 0, \quad +\pi/2 \leqslant \phi \leqslant 3\pi/2 \qquad \ldots \quad (5.4)$$

where

$$\phi_2 = \phi - \Omega/K + \pi/2.$$

and

$$\alpha = R.C.$$

Combining (5.3) and (5.4) the phase equation of such a system can be written as

$$\frac{d^2\psi}{dt^2} + 2b\,\frac{d\psi}{dt} \pm C\psi = 0, \qquad \ldots \quad (5.5)$$

Writing $\omega = d\psi/dt$, we have

$$\frac{d\omega}{d\psi} = 2b \mp C\psi/\omega. \qquad \ldots \quad (5.6)$$

As an example, we shall consider the case when $b = 0.5$ and $C = 1$. The 'phase-plane' plots for this case are shown in Fig. 9(a) and 9(b). Now when $\phi$ lies between

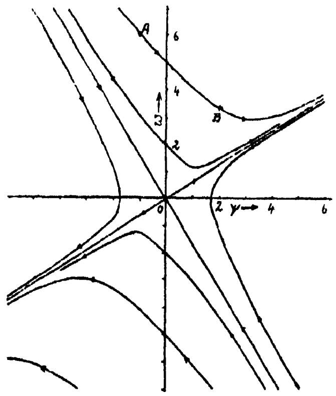

Fig. 9($a$) Phase-plane trajectories.

$$\frac{d\omega}{d\psi} = -1+\psi/\omega$$

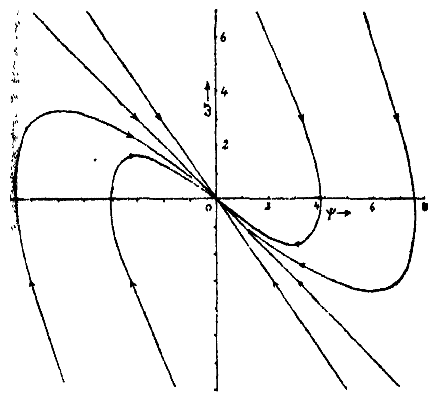

Fig. 9($b$) Phase-plane trajectories

$$\frac{d\omega}{d\psi} = -2-\psi/\omega$$

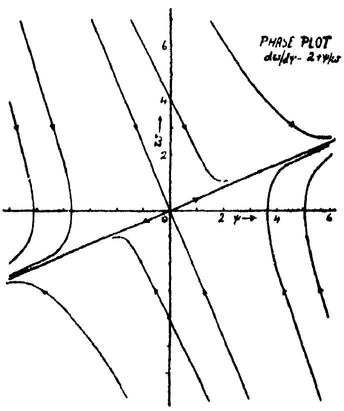


Fig. 9($c$) Phase-plane trajectories.

$$\frac{d\omega}{d\psi} = -2+\psi/\omega$$

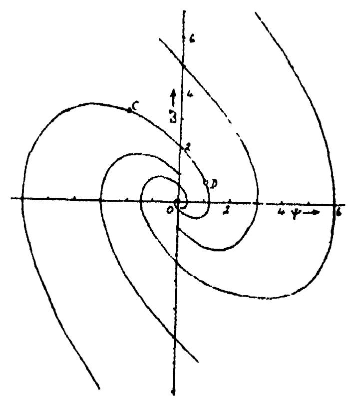

Fig. 9($d$) Phase-plane trajectories.

$$\frac{d\omega}{d\psi} = -1 - \frac{\psi}{\omega}$$

$-\pi/2$ and $+\pi\ 2$, $\psi$ lies between $-\pi/2-\Omega/K$ and $\pi/2-\Omega/K$ and similarly when $\phi$ lies between $\pi/2$ and $3\pi/2$, $\psi$ lies between $-\pi/2+\Omega/K$ and $\pi/2+\Omega/K$. On the assumption that $\Omega/K = 0.5$, Eq. (5.6) becomes

$$\frac{d\omega}{d\psi} = -1-\psi/\omega, -2.07 \leqslant \psi \leqslant 1.07 \qquad \dots \quad (5.7)$$

and

$$\frac{d\omega}{d\psi} = -1+\psi/\omega, \ -1.07 \leqslant \psi \leqslant 2.07 \qquad \dots \quad (5.8)$$

We may now consider a typical trajectory starting from $\psi = -1.07$, $\omega = 6$. The trajectory will lie in the region defined by (5.8), till at the end of the interval $AB$, $\psi$ becomes 2.07 and the instantaneous beat frequency $\omega$ becomes 3.3 (Fig. 9a). The next stretch of the trajectory starts from $\psi = -2.07$. Now at the end of this interval $CD$ (Fig. 9b) $\psi = 1.07$ and $\omega$ is seen to be reduced to 0.8. During the next successive cycles the value of the terminal $\omega$ continuously decreases (Figs. 9a and 9b) and ultimately the oscillator is pulled into synchronism. It thus requires a phase drift of about $4\pi$ radians for the oscillator to synchronise. As the ratio $\Omega/K$ is increased and is brought closer to unity, the oscillator takes more time to be synchronised. Now at the extreme case when $\Omega/K = \pi/2$ it can be shown that the instantaneous beat frequency at first decreases but it does so upto a certain limit. To study the effect of different values of the $RC$ product the trajectories for different values of '$b$' will have to be drawn and the motion of $(\omega, \psi)$ followed from the initial conditions through a required number of cycles till the equilibrium is attained.

Graphical construction, similar to those considered above, is possible for the filter network shown in Fig. 8(b). Here

$$G(p) = \frac{1+\alpha p}{1+\beta p}, \tag{5.9}$$

where $\alpha = xCR, \quad \beta = (1+x)CR.$

Now taking, for example, $K/\beta = 1$, we have

$$\frac{d\omega}{d\psi} = -\frac{1 \pm \alpha K}{\beta} \mp \psi/\omega. \tag{5.10}$$

The pull-in mechanism here will be clear from a study of the appropriate plots (see Fig. 9) for different values of the time constants $\alpha$ and $\beta$. The effect of different values of $\alpha$ and $\beta$ on the locking range and time will also be evident.

5.2. *APC circuit with Filter and a Sinusoidal Comparator* :

The governing equation of an APC circuit with a lowpass filter in the loop (Fig. 10) is given by the relation


Fig. 10. Block diagram of an APC system with filter.

$$\frac{d\phi}{dt} = \Omega - KG(p)\sin\phi, \qquad \dots \quad (5.11)$$

where $G(p)$ represents the gain function of the lowpass filter network normalised with respect to the maximum gain at zero frequency. As was pointed out in the discussion on the simple APC loop, the phase function contains not only the component at the fundamental beat frequency but also harmonic components. Now for a low pass filter it is logical to assume that the filter-transmission at the harmonic frequencies $2\omega$, $3\omega$ etc. is negligible. Then outside the pull-in range, the phase function (Cf. Eq. 4.7) takes the simple form

$$\phi = \omega t + \alpha + m \sin \omega t \quad \ldots \quad (5.12)$$

where $\omega$ is the beat frequency.

The steady state loop equations then become

$$\omega = \Omega + J_1(m) \frac{G_0 y}{\sqrt{\sin^2 \phi_1 + y^2 \cos^2 \phi_1}}, \quad \ldots \quad (5.13)$$

and $$\left[\frac{m}{J_0(m)+J_2(m)}\right]^2 \left(\frac{\omega \cos \phi_1}{G_1}\right)^2 + \left[\frac{m}{J_0(m)-J_2(m)}\right]^2 \left(\frac{\omega \sin \phi_1}{G_1}\right)^2 = K^2 \quad \ldots \quad (5.14)$$

where $$y = \frac{J_0(m)-J_2(m)}{J_0(m)+J_2(m)}$$

and $J_n(m)$ is Bessel's Function of order '$n$' and argument '$m$', and $\phi_1$ is the filter phase and $G_0$ and $G_1$ represent respectively d.c. and a.c. gains of the filter. These equations enable determination of $\alpha$, $\omega$ and $m$ for a given $\Omega$. Now if the Eqs. (5.13) and (5.14) can be simultaneously satisfied for any value of the beat frequency $\omega$ and a value of $m$ less than unity, the system will not lock and will show stationary beats. On the other hand, 'instability', leading to synchronisation, will set in if the value of $m$ demanded by the Eqs. (5.13) and (5.14) equals or exceeds unity.

The plots of the equations (5.13) and (5.14) for a simple $R$-$C$ network are shown in Fig. 11. With the help of this figure one can find the variation of $\omega$ and $m$

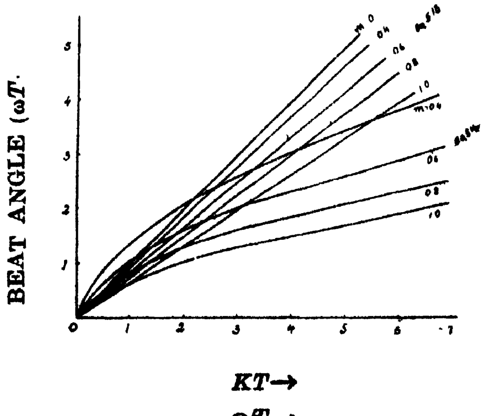


Fig. 11. Graph illustrating the evaluation of '$m$' and '$\omega$' using Eqs. 5.13 and 5.14.

with time till the limiting value of the beat frequency corresponding to $m = 1$ is reached, and also whether the system will synchronise or not.

## LOCKING RANGE AND LOCKING TIME

An APC loop is characterised by the following parameters :(i) the locking range, (ii) the locking time and (iii) the noise bandwidth (see Appendix for an expression for the noise-bandwidth). The locking time can be thought of as composed of frequency pulling time and phase pulling time. Phase pulling time is the time taken by the loop to annul the discrepancy in phase between the input and the output, if the initial difference frequency is very small. Frequency pulling time, on the other hand, is the time required for equalisation of the frequency of the input and the output brought about by a gradual increase of the steady discriminator voltage. It should be remarked that the frequency pulling time is considerably larger than the phase pulling time. (The latter has been briefly analysed in the Appendix).

During the period of phase pulling, one may consider the APC loop as a semi-direct-current loop, while during the period of frequency pulling the loop is a combined A.C. and DC. one, and the A.C. gain characteristic assumes a significant role. Now there may arise two distinct cases—(a) one in which the limiting high frequency gain is zero and (b) the other in which it is finite. Whatever the case may be, in the long time process of frequency pulling there will be a steady drift of the difference frequency per beat cycle. The amount of this drift is determined by the initial difference frequency and the gain of the open loop at the beat frequency. The situation is best analysed by considering the phenomenon in a sequence of beat cycles. The phenomenon of frequency pulling in APC loops with two different types of filters mentioned above will be studied in some details in this section and expressions for locking range and time will then be derived.

### 6.1. *Filter with Finite A.C. Gain* :

We shall assume that the gain function $G(P)$ may be separated into two parts-high frequency gain $G_H(P)$ and a low frequency portion $G_L(p)$, i.e.,

$$G(p) = G_H(p)+G_L(p). \qquad \dots \quad (6.1)$$

Further it will be assumed that the high frequency gain is considerably smaller than the low frequency gain. If the initial difference frequency $\Omega$ is lower than $KG_H$, the phase will attain a steady value in a fairly short period. On the other hand, if $\Omega > KG_H$ the system may still attain the equilibrium, but the time will be considerably longer. In the period of a beat cycle, the beat frequency may be considered to be constant. There will, however, be a steady drift of the value of this beat frequency over a number of such cycles. Now

$$\frac{d\phi}{dt} = (\Omega - KG_L \sin\phi) - KG_H \sin\phi. \qquad \dots \quad (6.2)$$

If the time constants involved in locking are long compared with a period of a beat cycle, we can write,

$$K G_H \overline{\sin\phi} = -p\phi + \Omega - K\overline{G_L \sin\phi},$$

$$\text{i.e., } K\overline{G_H \sin\phi} = -\omega_\beta + \Omega - \overline{KG_L \sin\phi} \qquad \ldots \quad (6.3)$$

Now putting $\omega_1 = \Omega - KG_L \sin\phi,$

or, $$\bar{\omega}_1 = \Omega - K\overline{G_L \sin\phi}. \qquad \ldots \quad (6.4)$$

We have from (6.3)

$$\omega_\beta = \sqrt{\bar{\omega}_1^2 - (KG_H)^2} + \bar{\omega}_1. \qquad \ldots \quad (6.5)$$

So that $$KG_H \overline{\sin\phi} = -\sqrt{\bar{\omega}_1^2 - (KG_H)^2} + \bar{\omega}_1. \qquad \ldots \quad (6.6)$$

This equation gives us the value of the effective difference frequency at a given time. Equations (6.4) and (6.6) yield

$$\bar{\omega}_1 = \Omega + \frac{G_L}{G_H}\left[\sqrt{\bar{\omega}_1^2 - (KG_H)^2} - \bar{\omega}_1\right]. \qquad \ldots \quad (6.7)$$

Equation (6.7) can be written as a differential equation in $\bar{\omega}_1$, which can then be solved for evaluating frequency pulling time (Richman, 1954). For example, if

$$G(p) = m + \frac{1-m}{1 + p\frac{y}{m} t_c},$$

where $y = x\,T/t_c$, $t_e = 1/\mu\beta = 1/K$ and $m$ is ratio of a.c. gain to d.c. gain of the filter. Now replacing $\sqrt{\bar{\omega}_1^2 - (mK)^2}$ by the approximate relation $\dfrac{\bar{\omega}_1(\bar{\omega}_1 - mK)}{\bar{\omega}_1 - 0.84mK}$ and putting $\bar{\omega}_1/mK = \rho$, $\Omega/mK = \rho_0$ we have ultimately the following relations for the locking time and range :

$$\frac{mT_F}{yt_c} = \left[-\frac{1}{2}\log_e(\rho^2 - 2b\rho + 0.84\rho_0) - \frac{2b - 0.84}{2} \cdot \frac{1}{\sqrt{0.84\rho_0 - b^2}} \tan^{-1}\frac{\rho - b}{\sqrt{0.84\rho_0^2 - b^2}}\right]_{\rho_0}^{1} \qquad \ldots \quad (6.8)$$

and $$\Omega/K \geqslant 0.16 + 0.84m, \qquad \ldots \quad (6.9)$$

where $$2b = \rho_0 + \frac{m - 0.16}{m}$$

6.2. *Filter with Negligible High-Frequency Gain* :

There may be systems in which there is no high frequency gain or the high frequency gain falls rapidly with frequency. An example is provided by a simple $R$-$C$ filter (Rey, 1960). In this case, the loop equation reduces to

$$\frac{d\phi}{dt} = \Omega - \frac{K}{1+pT} \sin\phi, \qquad \text{... (6.10)}$$

or,

$$T\frac{d^2\phi}{dt^2} + \frac{d\phi}{dt} + K\sin\phi = \Omega, \qquad \text{... (6.11)}$$

Where $T$ is the time constant. We may assume that

$$\dot{\phi} = \omega t + \alpha + m\sin(\omega t - \beta) \qquad \text{... (6.12)}$$

Hence from these two equations we have

$$T\frac{d\omega}{dt} + \omega - KJ_1(m)\sin\beta = \Omega, \qquad \text{... (6.13)}$$

$$T\frac{d}{dt}(m\omega) + m\left(\omega - \frac{d\beta}{dt}\right) + K(J_0 + J_2)\sin\beta = 0, \qquad \text{... (6.14)}$$

$$-T\left(\omega - \frac{d\beta}{dt}\right)\omega + \frac{dm}{dt} + K(J_0 - J_2)\cos\beta = 0. \qquad \text{... (6.15)}$$

To a first order of approximation

$$K(J_0 + J_2)\sin\beta \simeq -m\left(\omega - \frac{d\beta}{dt}\right), \qquad \text{... (6.16)}$$

and

$$K(J_0 - J_2)\cos\beta \simeq T\left(\omega - \frac{d\beta}{dt}\right). \qquad \text{... (6.17)}$$

Therefore from the Eqs. (6.12), (6.16) and (6.17)

$$T\frac{d\omega}{dt} + \omega - K\frac{1}{2\sqrt{1+\omega^2T^2}} = \Omega. \qquad \text{... (6.18)}$$

from which locking range and time can be found out. Again from above, we have

$$m\omega = KG.J_0(m). \qquad \text{... (6.19)}$$

The locking range is given by

$$\Omega_{10} = KG\left(1 + \frac{\cos\phi_F}{2G}\right), \qquad \text{... (6.20)}$$

where $\phi_F$ denotes the filter-phase angle.

The locking time can be found by evaluating

$$T_F = T \int_{\omega_{max}}^{KG} \frac{d\omega}{\Omega - F(\omega)} \tag{6.21}$$

where

$$F(\omega) = \omega - \frac{K}{2} \frac{1}{\sqrt{1+\omega^2 T^2}}$$

$$= \omega - \frac{K}{2} \, | G(\omega) | .$$

It should be mentioned that equations (6.19), (6.20) and (6.21) will apply for any APC loop with a filter having negligible asymptotic gain.

6.3. *Derivation of Simple Expressions for Locking Range and Time :*

Locking range and time can also be estimated from the following simple treatment. We shall here assume that the instantaneous phase contains : (i) a very slowly varying component $\alpha$, corresponding to a slow change of the averge discriminator voltage, (ii) a component varying linearly with time at the rate of the beat angular frequency and (iii) a component periodic at the beat frequency. If we further assume that the significant component of the periodic part is the fundamental, the following equations are obtained to a first order of approximation:

$$\frac{d\alpha}{dt} = \Omega - \omega + KG_0 J_1(m) \sin \alpha, \tag{6.22}$$

$$\frac{dm}{dt} = -KG[(J_0 - J_2) \cos \alpha \cos \phi - (J_0 + J_2) \sin \alpha \sin \phi]; \tag{6.23}$$

and $$m\omega = -KG[J_0 \sin (\alpha + \phi) + J_2 \sin (\alpha - \phi)]. \tag{6.24}$$

Using these equations (Eq. 6.22 to 6.24), one can derive an approximate equation for the average drift in frequency

$$\frac{d\alpha}{dt} = (\Omega - 0.8K\bar{G}) - 0.2KG_0 \sin \alpha \tag{6.25}$$

where $\bar{G}$ stands for the average value of the network gain over the range—d.c. to the initial difference frequency $\Omega$. The weighted averaging involved in this derivation has been done with due regard to the ranges over which the quantities vary. It is to be noted, for example, that the value of '$m$' will be initially small, but after a while will approach unity.

Eq. (6.25) is identical in form with Eq. (4.3) and gives both the locking range and time. Thus assuming $G_0 = 1$, the initial difference frequency is given by

$$\Omega = K\bar{G} \quad \ldots \tag{6.26}$$

The locking time can be found by evaluating

$$T_F = \int_{\alpha_i}^{\alpha_f} \frac{d\alpha}{\Omega_0 - K_0 \sin\alpha} \qquad \dots \quad (6.27)$$

$$\Omega_0 = \Omega - 0.8KG, \quad K_0 = .2KG_0$$

where $\alpha_i$ and $\alpha_f$ are the initial and final values of $\alpha$.

The initial value of $\alpha$ can be found by putting $d\alpha/dt = 0$, $dm/dt = 0$ in Eqs. (6.22) to (6.24). An working approximate value for $\alpha_i$ is $[\pi/2 - \phi(\Omega)]$. The final value is obviously equal to $\sin^{-1}(\Omega/K)$. Using Eq. (6.27) it is now possible to find the locking time for a given difference frequency.

It will appear that the knowledge of the average again $\bar{G}$ and the network phase $\phi(\Omega)$ at $\Omega$ is sufficient to obtain reasonably accurate estimates of the locking time and range (see Sec. 7). The value of $\bar{G}$ in Eqs. (6.26) and (6.25) can be found out either analytically or graphically. Analytical solutions are possible only in simple cases and one will have to take recourse to graphical techniques in cases when the network gain function contains a number of poles and zeros.

Graphical constructions for the filters (Figs. 8a and 8b) have been shown in Figs. 12 and 13 from which the evaluation of locking ratio $(\Omega/K)$ can be accomplished with the help of equation (6.26).

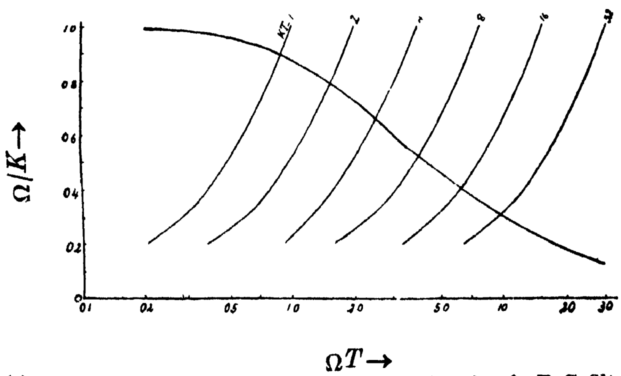


Fig. 12. Graphical determination of locking ratio for the simple R-C filter of Fig. 8($a$)

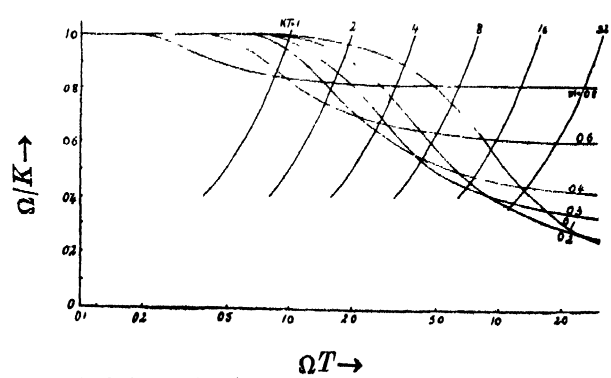


Fig. 13. Graphical determination of locking ratio for the filter of Fig. 8($b$).

## EXPERIMENTAL SET-UP AND RESULTS

In this section we shall first describe the experimental set-up. This will be followed by a discussion of the results obtained and comparison of these with the results of the analysis presented in section 4.

Fig. 14 shows the experimental set-up for making measurements of the locking range ($\Omega$) for two types of filters—one having negligible high frequency

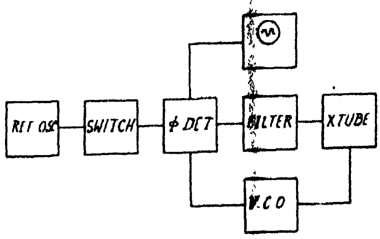


Fig. 14. Experimental set-up.

gain and the other having finite high frequency gain-to check the theoretically derived results. The detailed circuit diagram is shown in Fig. 15. The phase shift of 90° repuired between the grid and the plate voltages in the reactance tube (4) has been achieved by means of a two stage R–C phase-shifter ($R_{13}C_{13}R_{14}C_{14}$)

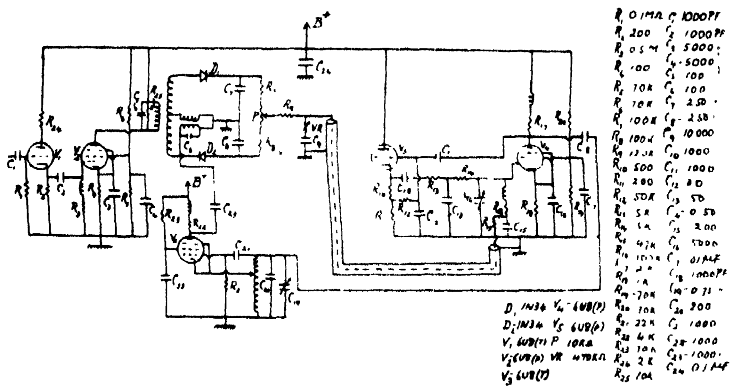

Fig. 15. Detailed circuit diagram of the system.

provided with a trimmer to effect accurate adjustments. This arrangement was adopted as it ensures negligible conductance loading of the reactance tube ($V_4$). The oscillator to be synchronised is electron coupled tuned grid Hartely type ($V_5$). The phase detector used is of the balanced demodulator type. Its output is the difference between the voltages $|E_c+E_i|$ and $|E_c-E_i|$ where $E_c$ an $E_i$ are proportional respectively to the oscillator and input synchronising voltages. For a successful operation of the circuit it is necessary to guarantee that the ratio of the magnitudes of the two voltages never approaches unity. The centre frequency of the oscillator and the input amplifier need be carefully adjusted to the same value. Further, the input amplifier for feeding the phase detector should have a flat top characteristics. Presence of a dip anywhere in its response characteristics is likely to produce spurious effects and sometimes a type of oscillation. It is necessary to ensure that the time constant of the detector circuit be such that the *r-f* filtering is adequate, yet is small enough as not to interact with the filter time constants. In fact the latter consideration sets the lowest limit to the minimum filter time constant usable. It has been found necessary to incorporate an *r.f.* choke in the path from the detector-filter to the grid of the reactance tube to provide effective *r.f.* decoupling. It should also be mentioned that large variations of resistance in the d.c. grid circuit should be avoided as it affects the sensitivity of the X-tube. The value of the limiting angular frequency of synchronisation (K) obtaining in the circuit described is 10 kiloradian/sec.

In Fig. 16 the experimental values of the locking ratio ($\Omega/K$) have been plotted against the product of the limit of synchronising frepuency in radian per sec and the filter time constant of the network of Fig. 8a. The variation of time constant was here effected by changing the value of the capacitance $C$. It will be found that the experimental values are in close agreement with the corresponding values computed from Eq. (6.26) which are also shown in Fig. 16. The values of locking

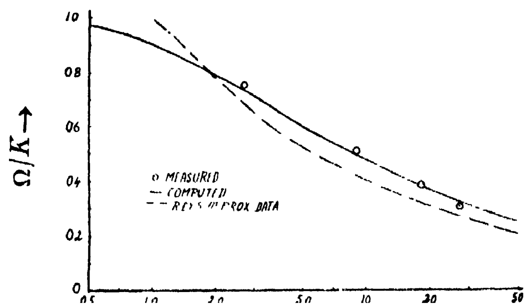


Fig. 16. Pull-in performance of the system with filter of Fig. 8(*a*).

ratio obtained by computations using Eq. (6.20) have been found to be close to but a little less than the experimental values.

In the APC circuit using the filter of Fig. 8b, the value of the ratio of the a.c./d.c. gain was varied by changing the value of XR and the time constant by

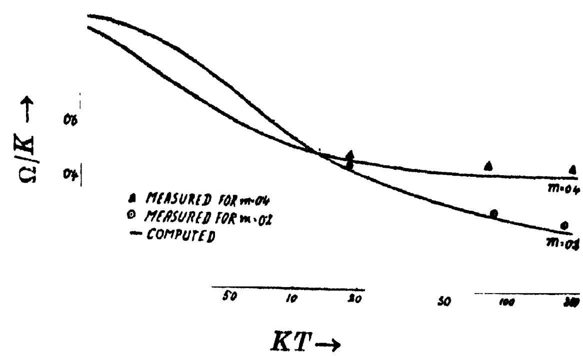


Fig. 17. Pull-in performance of the system with filter of Fig. 8(*b*).

changing the capacitance $C$. Fig. 17 shows the values of the locking ratio determined experimentally as well as those computed from Eq. (6.26) for this case. (Photographs showing transient pull-in for the filter of Fig. 8(b) are given in

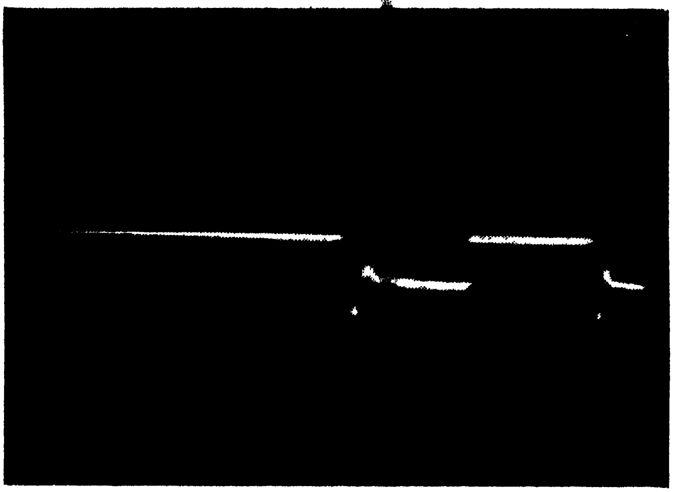

Fig. 18. (*a*) Photograph showing pull-in time for an APC circuit with the filter of Fig. 8(*b*) when the initial difference frequency is: 0.5 Kc/s below the centre frequency.

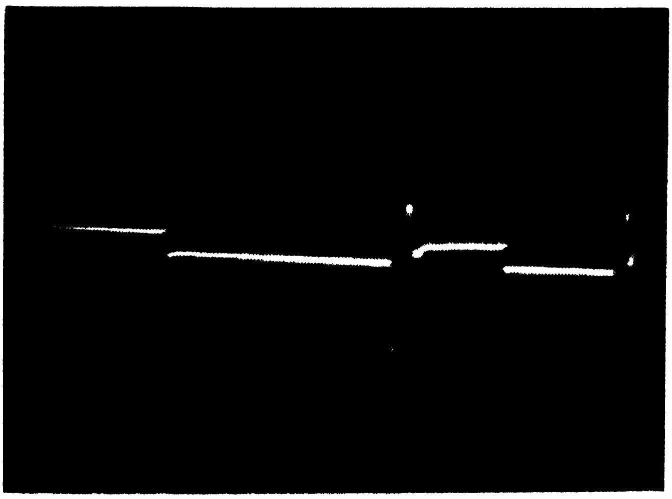

Fig. 18. (*b*) 0.5 Kc/s above the centre frequency.

Fig. 18). Here also the agreement is fairly close. If, however, Eq. (6.9) is used for computing the locking ratio, it will be found that the computed values agree with the experimental values for large values of $T$ only. The discrepancy at low values of $T$ arises from the fact that the difference between the high and the low frequency time constants is then small.

It can be concluded from the above results that Eq. (6.26) provides a simple

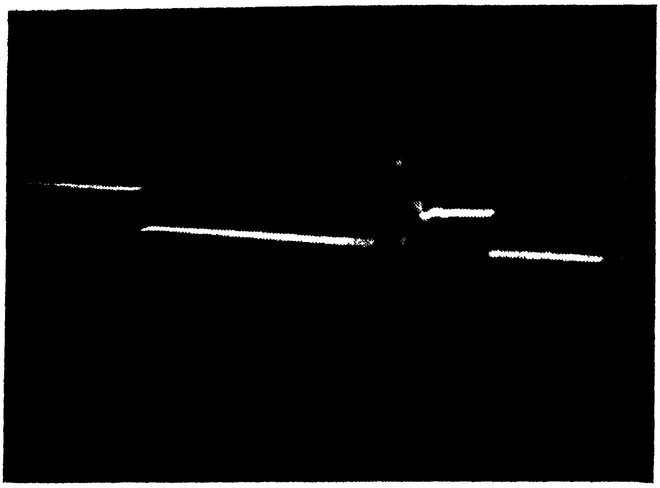

Fig. 18. (c) 1.0 Kc/s above the centre frequency.

yet fairly accurate relation for calculating the locking range and time of APC systems with filters of a wide class.

## CONCLUDING REMARKS

The locking phenomenon in APC circuits with two different filter networks has been analysed for continuous synchronising signal at the input and their performance studied experimentally. The cases of interrupted wave synchronisation and the synchronisation with a signal having low S/N ratio will be considered in a future communication.

## ACKNOWLEDGMENT

The authors wish to thank Prof. J. N. Bhar, D.Sc., F.N.I.,for his kind interest and encouragement.

## APPENDIX

A.1. *Response to a step-change in '$\Omega$' and phase Locking Range and Time*

We shall here consider the response of an APC circuit to a step change in the angular frequency '$\Omega$' applied to the input. If the instantaneous phase difference between the external input and the reference input is sufficiently small, then the governing equation of the loop is :

$$\frac{d\phi}{dt} \simeq \Omega - \frac{2}{\pi} \cdot KG(p) \cdot \phi. \tag{A.1}$$

With this approximation one can readily find the minimum value of $\phi$ so that the system never loses cycles by determining the condition that the resultant phase on excitation by a step in frequency never exceeds $\pm\pi/2$. For the purpose one has only to find the amount of overshoot $\gamma$ in the step response of

$$F(p) = \frac{\Omega}{p[p+\frac{2}{\pi} K \cdot G(p)]} \quad \dots \quad (A.2)$$

The value of the steady state phase $\phi_s = \pi/2 \cdot \frac{\Omega}{K}$. Hence the required condition is seen to be,

$$\frac{\pi}{2} \Omega (1+\gamma) < K\pi/2$$

i.e.,

$$\Omega (1+\gamma) < K. \quad \dots \quad (A.3)$$

If this condition is satisfied one can find the phase locking time by calculating the time required for settling to a value within five per cent of the difference between the initial and final phases. Let us assume that a sudden step change in frequency is applied at $t = 0$, when the initial conditions are $\phi = 0$, and $d\phi/dt = 0$. Then Eq. (A.1) reduces to

$$[1+(1+x)pT]p \cdot \bar{\phi}(p) + \frac{2}{\pi} \cdot K \cdot (1+xpT)\phi(p) = \frac{\Omega}{p} [1+(1+x)pT] \dots \quad (A.4)$$

Substituting

$$T_1 = \frac{(1+x)T}{1+\frac{2}{\pi} xKT},$$

and

$$K_1 = \frac{K}{1+\frac{2}{\pi} xKT} \quad \dots \quad (A.5)$$

and comparing (A. 4) with (A. 5) we have

$$\phi(p) = \frac{\Omega}{p} \cdot T_1 \left[p+\frac{1}{(1+x)T}\right] \cdot \frac{1}{p^2T_1+p+2/\pi\, K_1} \quad \dots \quad (A.6)$$

or,

$$\phi(p) = \frac{\Omega T_1\left[p+\frac{1}{(1+x)T}\right]}{p[p+\alpha+j\beta][p+\alpha-j\beta]} \quad (A.7)$$

where $(\alpha \pm j\beta)$ are the roots of the polynomial $p^2T_1+p+2/\pi\, K_1 = 0$. Hence the variation of phase function with time is given by

$$\phi(t) = \frac{\Omega\, T_1}{\alpha^2+\beta^2}\left[\frac{1}{(1+x)T} - \frac{\epsilon^{-\alpha t}}{\beta} \cdot \mathrm{A} \sin(\beta t+\psi)\right], \quad (A.8)$$

where,

$$A = \left[ \left\{ \frac{\alpha}{(1+x)T} - (\alpha^2+\beta^2) \right\}^2 + \left( \frac{\beta}{(1+x)T} \right)^2 \right]^{\frac{1}{2}} \quad \ldots \quad (A.9)$$

and

$$\psi = \tan^{-1} \frac{\dfrac{\beta}{(1+x)T}}{\dfrac{\alpha}{(1+x)T} - (\alpha^2+\beta^2)} \quad \ldots \quad (A.10)$$

From Eq. (A.8) the percentage overshoot will be found to be given by

$$\frac{\phi_m - \phi_S}{\phi} = \frac{A(1+x)T}{\beta} \cdot \sin\theta \cdot \exp\left[-(\pi+\theta-\psi)\cot\theta\right] \quad \ldots \quad (A.11)$$

where $\phi_m$ and $\phi_S$ are the maximum and steady state phase shifts respectively and

$$\theta = \tan^{-1}(\beta/\alpha) \quad \ldots \quad (A.12)$$

From Eqs. (A.3) and (A.11) the locking ratio is seen to be given by the approximate relation :

$$\frac{\Omega}{K} \simeq \frac{1}{1+q \cdot y} \quad \ldots \quad (A.13)$$

where

$$q = \frac{A}{\beta} \sin\theta \cdot \exp\left[-\pi+\theta-\psi\right) \cot\theta] \quad \ldots \quad (A.14)$$

$$y = (1+x)T \quad \ldots \quad (A.15)$$

The instants when the phase equals $\phi_s$ can be found from Eq. (A.8). Thus $t_n = \dfrac{n\pi-\psi}{\beta}$. An approximate value for the phase settling time can be written as

$$T_p \simeq \frac{1}{3\alpha} \cdot \log\left[\frac{A}{\beta} \cdot y\right] \quad \ldots \quad (A.16)$$

A.2. *Noise Bandwidth* :

Noise bandwidth can be defined as

$$B_n = \int^{\infty} |G(\omega)|^2 df \quad \ldots \quad (A.17)$$

where $G(\omega)$ is the normalised closed loop transfer function. (A.17) can also be written as

$$B_n = \frac{1}{2\pi j} \int_0^{\infty} G(p)\, G(-p) dp \quad (A.18)$$

Now for the lag network, we have

$$f(p) = \frac{1+xpT}{1+(1+x)pT} \quad \ldots \quad (A.19)$$

and
$$G(p) = \frac{Kf(p)}{p+Kf(p)} \qquad \ldots \quad (A.20)$$

Hence
$$B_n = \frac{K}{4} \cdot \frac{(\alpha+xKT)}{\alpha(1+xKT)} \qquad \ldots \quad (A.21)$$

where
$$\alpha = (1+1/x). \qquad \ldots \quad (A.22)$$

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